Generalized Market Equilibrium: ”Stable” CAPM

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Abstract. Our main purpose in this paper is to derive the generalized equilibrium relationship between risk and return under the assumption that the asset returns follow a joint symmetric α stable distribution, with $1 < \alpha < 2$. We show that equilibrium rates of return on all risky assets are functions of their **covariation** with the market portfolio. The "stable" CAPM highlights a new measure of the quantity of risk which may be interpreted as a "generalized beta coefficient".

1. Introduction

Most economic and financial phenomena are modeled using probability distributions with finite variance. In particular, Capital Asset Pricing Model (CAPM) theory has been developed by many authors in a Gaussian framework (see e.g. Sharpe [Sha63, Sha64], Treynor [Tre61], Mossin [Mos66], Lintner [Lin65], Black [Bla72], Fama-MacBeth [FM73] and Blume-Freind [BF73]). However the assumption of Gaussianity is in general not verified empirically. It has been shown in many studies that asset returns exhibit a fat tail in their empirical distributions. Mandelbrot [Man63] and Fama [Fam65] proved that empirical distributions of asset returns such as stocks, foreign currencies, etc ... conform better to stable distributions than to the normal distribution. Asset returns may thus be naturally modeled by random variables defined on a complete probability space $L^p(\Omega, \mathcal{F}, P)$ with $1 < p < 2$. This implies that the mean is assumed to be finite but it is not necessary so for the variance. In this context, the classical mean-variance approach for optimal portfolios selection does not make sense. Its use leads to discard important information about the risk structure of different investment portfolios and may result in underestimating the quantity of risk and the risk premium.

Several authors have thus proposed to use stable Paretian distributions to model returns on securities. This raises the following questions :

(1) how should one measure the dependence between returns ?
(2) how can one characterize financial risk ?

Such questions are studied by Press [Pre72], Lee-Rachev-Samorodnitsky [LRS90], Rachev-Xin [RX93] and Samorodnitsky-Taqqu [ST94].

Following these investigations, we shall extend in this paper the concept of market equilibrium in order to develop the CAPM when asset returns have a joint stable distribution. It will be shown that the equilibrium rates of return on all risky assets are functions of their **covariation** with the market portfolio. This is a natural generalization of the results obtained in the "classical" case, where the asset returns are modeled with Gaussian distributions.

We mention here that Arad [Ara75] already considered the problem of the CAPM under the hypothesis of stable distribution in a particular case, "the stable index model", which assumes a special structure of dependence (all securities depend on a common factor or on a number of common terms).

In section 2 we briefly state some well known definitions and properties of multivariate stable distributions. In sections 3 and 4 we recall some useful results concerning the measure of dependence and financial risk when one deals with heavy tailed multivariate distributions. In section 5 we shall define the generalized market line in the mean-scale space for a given α (as defined below) and derive the CAPM under the following assumptions :

(1) All investors have homogeneous expectations about asset returns.
(2) The common probability distribution of asset returns is joint Levy-stable and satisfies a symmetry condition (see section 4).
(3) All investors are risk averse.
(4) An investor may borrow or lend unlimited amounts at the risk free rate.
(5) There are no market imperfections.

In the last section we present an experimental study and results on real data, and draw some practical consequences of the generalized CAPM.

2. Definitions and properties

For the definition of stable multivariate distributions we essentially make reference to Rachev-Xin [RX93] and Samorodnitsky-Taqqu [ST94].

Definition 1.

A random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leq 2$, $\gamma \geq 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$ such that its characteristic function has the following form

$$\Psi_X(t) = \exp\left\{i\mu t - \gamma^\alpha |t|^\alpha \left(1 - i\beta\ sign(t) W(\alpha, t)\right)\right\}, t \in \mathbb{R} \tag{1}$$

where

$$W(\alpha, t) = \begin{cases} \tan \frac{\pi\alpha}{2} & if\ \alpha \neq 1 \\ -\frac{2}{\pi} \log |t| & if\ \alpha = 1 \end{cases} \tag{2}$$

The four parameters characterizing a stable random variable are sometimes referred to as :

- The characteristic exponent α, $0 < \alpha \leq 2$. It describes the shape of the distribution or the degree to which it is long tailed.
- The index of skewness β, $\beta \in [-1, 1]$.
- The location parameter μ, $\mu \in \mathbb{R}$.
- The scale parameter γ, $\gamma \in \mathbb{R}^+$.

Notation 1.

$X \stackrel{d}{=} S_{\alpha,\beta}(\gamma, \mu)$ indicates that X follows a stable distribution with parameters α, β, γ and μ (as defined above).

Property 1.

$X \stackrel{d}{=} S_{\alpha,\beta}(\gamma, \mu)$ is symmetric if and only if $\beta = 0$ and $\mu = 0$. It is symmetric about μ if and only if $\beta = 0$.

Notation 2.

$X \sim S\alpha S$ indicates that X follows a stable distribution $S_{\alpha,0}(\gamma, 0)$

From now on, X will denote a d-dimensional random vector.

Definition 2.

X follows an α-stable multivariate distribution ($0 < \alpha < 2$) if there exists a finite measure Γ on the unit sphere S_d of $\mathbb{R}^d$ ($S_d = \{s, \ ||s|| = 1\}$), and a shift vector $\mu^0 \in \mathbb{R}^d$ such that:

$$\Psi_X(\lambda) = \exp\left\{ i(\lambda, \mu^0) - \int_{S_d} |(\lambda, s)|^\alpha \left(1 - i \ sign((\lambda, s)) W(\alpha, s, \lambda)\right) \Gamma(ds) \right\}, \lambda \in \mathbb{R}^d \tag{3}$$

where (.,.) denotes the inner product and

$$W(\alpha, s, \lambda) = \begin{cases} \tan \frac{\pi\alpha}{2} & \text{if } \alpha \neq 1 \\ -\frac{2}{\pi} \log |(\lambda, s)| & \text{if } \alpha = 1 \end{cases} \tag{4}$$

The pair (Γ, μ^0) is unique.

The measure Γ is called the spectral measure of the α-stable random vector X. It replaces both the scale and skewness parameter that enter in the description of the univariate stable distribution.

As in the case of random Gaussian vectors, any linear combination of the components of an α-stable vector is an α-stable random variable:

Property 2.

Let X be an α-stable random vector with characteristic function given by (3) and let $Y = (\theta, X)$, $\theta \in \mathbb{R}^d$. Then Y has a univariate α-stable distribution $S_{\alpha,\beta_\theta}(\gamma_\theta, \mu_\theta)$, where

$$\gamma_\theta = \left(\int_{S_d} |(\theta, s)|^\alpha \Gamma(ds)\right)^{\frac{1}{\alpha}} \tag{5}$$

$$\beta_\theta = \frac{\int_{S_d} |(\theta, s)|^\alpha \, sign((\theta, s))\Gamma(ds)}{\int_{S_d} |(\theta, s)|^\alpha \Gamma(ds)} \tag{6}$$

$$\mu_\theta = \begin{cases} (\theta, \mu^0) & \text{if } \alpha \neq 1 \\ (\theta, \mu^0) - \frac{2}{\pi} \int_{S_d} (\theta, s) \log |(\theta, s)| \Gamma(ds) & \text{if } \alpha = 1 \end{cases} \tag{7}$$

Proof. *See Samorodnitsky-Taqqu* [ST94].

Property 3.

X is a symmetric α-stable vector in $\mathbb{R}^d$, $0 < \alpha < 2$ if and only if there exists a unique symmetric finite measure Γ on the unit sphere S^d such that

$$E \exp i(\lambda, X) = \exp\left\{-\int_{S_d} |(\lambda, s)|^\alpha \Gamma(ds)\right\}, \ \lambda \in \mathbb{R}^d \tag{8}$$

Thus X is symmetric if and only if the shift vector $\mu^0 = 0$ and the spectral measure Γ is symmetric.

3. Dependence measure

The dependence structure of a Gaussian random vector ($\alpha = 2$) is completely specified by its autocovariance function. There is no such simple description when $\alpha < 2$, because covariance does not exist. But the notions of **covariation** (when

$\alpha \geq 1$) and **codifference** (when $0 < \alpha \leq 2$) prove to be very natural measures of dependence when one deals with heavy tailed multivariate distributions such as the stable one.

3.1. Covariation.

The notion of "covariation" is due to Miller [Mil78]. It is designed to replace the covariance when $1 < \alpha \leq 2$. We list some of the useful properties of covariation (see Samorodnitsky-Taqqu [ST94] for the proofs).

Definition 3.

Let X_1 and X_2 be jointly $S\alpha S$ with $\alpha > 1$ and let Γ be the spectral measure of the random vector (X_1, X_2). The covariation of X_1 on X_2 is the real number

$$[X_1, X_2]_\alpha = \int_{S_2} s_1 s_2^{<\alpha-1>} \Gamma(ds) \tag{9}$$

where $a^{<\alpha-1>}$ is the "signed power" defined by $a^{<\alpha-1>} = |a|^{\alpha-1} sign(a)$.

Properties.

(1) $[X_1, X_2]_2 = \frac{Cov(X_1, X_2)}{2}$
(2) $[X_1, X_2]_\alpha \neq [X_2, X_1]_\alpha$ *in general*
(3) $[aX_1, bX_2]_\alpha = ab^{<\alpha-1>}[X_1, X_2]_\alpha$ *for every a, $b \in \mathbb{R}$*
(4) *the covariation is additive in the first argument i.e. for (X_1, X_2, X_3) jointly $S\alpha S$ and for every a, $b \in \mathbb{R}$*

$$[aX_1 + bX_2, X_3]_\alpha = a[X_1, X_3]_\alpha + b[X_2, X_3]_\alpha$$

(5) *the covariation is not additive in general in its second argument i.e. for (X_1, X_2, X_3) jointly $S\alpha S$*

$$[X_1, X_2 + X_3]_\alpha \neq [X_1, X_2]_\alpha + [X_1, X_3]_\alpha$$

(6) *If X_1 and X_2 are jointly $S\alpha S$ and independent then $[X_1, X_2]_\alpha = 0$*
(7) *the covariation induces a norm $||\ ||_\alpha$ on the linear space S_α of jointly $S\alpha S$ ($\alpha > 1$) random variables. The norm $||\ ||_\alpha$ is defined for every $X_1 \in S_\alpha$ by $||X_1||_\alpha = ([X_1, X_1]_\alpha)^{\frac{1}{\alpha}} = \gamma_{X_1}$, where γ_{X_1} is the scale parameter of X_1.*

3.2. Codifference.

Although we are not going to use the notion of codifference in this work, we give some basic facts related to it because it could well be useful for solving the kind of problems we deal with.

The codifference function is derived from the difference between the joint characteristic function and the product of the marginal characteristic functions. It was first introduced by Astrauskas [Ast83]. and is defined for all $0 < \alpha \leq 2$.

Definition 4.

The codifference of two jointly $S\alpha S$ random variables X and Y equals :

$$\tau_{X,Y} = ||X||_\alpha^\alpha + ||Y||_\alpha^\alpha - ||X-Y||_\alpha^\alpha \tag{10}$$

where $||X||_\alpha^\alpha$, $||Y||_\alpha^\alpha$ and $||X-Y||_\alpha^\alpha$ denote respectively the scale parameters of X, Y and $X - Y$.

Properties.

(1) $\tau_{X,Y} = \tau_{Y,X}$

(2) *if* $\alpha = 2$ *then* $\tau_{X,Y} = Cov(X,Y)$

(3)
- *if X and Y are independent then* $\tau_{X,Y} = 0$
- *if $\tau_{X,Y} = 0$ and $0 < \alpha < 1$ then X and Y are independent*

(4)
- $\tau_{X,Y} \leq ||X||_\alpha^\alpha + ||Y||_\alpha^\alpha$
- $$\tau_{X,Y} \geq \begin{cases} 0 & \text{if } 0 < \alpha \leq 1 \\ (1-2^{\alpha-1})(||X||_\alpha^\alpha + ||Y||_\alpha^\alpha) & \text{if } 1 \leq \alpha \leq 2 \end{cases}$$

(5) *let $(X_1,\ ...,\ X_d)$ be a $S\alpha S$ random vector.*
Then the matrix $\Sigma = (\tau_{X_i,X_j},\ i,\ j = 1,\ ...,\ d)$ is non-negative definite.

For more facts and proofs of the properties listed above see Astrauskas [Ast83], Levy-Taqqu [LT91] and Samorodnitsky-Taqqu [ST94].

4. "Stable" risk measure and efficient set

When it exists, the variance is the statistic most frequently used to measure the risk. But we should note that there are other statistics which, in some situations, may be more appropriate : for instance the range, the semi-interquantile range, the semivariance and the mean absolute deviation have been considered (see Copeland-Weston [CW83]).

The Gaussian assumption leads to an optimal strategy depending on the means and variances of portfolios returns. However, when the distribution is α-stable ($1 < \alpha < 2$), the second moment does not exist, and an approach based on an empirical mean-variance computation discards important information about the risk structure of different investment opportunities. It can lead to the selection of non-optimal investment portfolios. In our context, the measure of risk will be the scale parameter of an appropriate multivariate symmetric stable distribution.

Let R be the vector of considered asset returns and $E(R) = \mu^0$. We assume that $R - \mu^0$ follows a $S\alpha S$ law with $\alpha > 1$. Press [Pre72] give several reasons for the assumption that $\alpha > 1$:

(1) for an investment setting, it is convenient to be able to speak of "expected returns";
(2) this assumption is in general confirmed by empirical evidence;
(3) although the distributions depart from Normality, they do not deviate "too much".

Finally, the symmetry assumption allows positive and negative price changes to be weighted in the same way.

The investors' preferences can then be represented by a utility function defined over the mean and the scale of a portfolio return $R_p = \sum_{i=1}^{d} \theta_i R_i$, where R_i is the return of the asset i and θ_i is the amount invested in the asset i.

$$E(R_p) = (\theta, \mu^0) \tag{11}$$

$$||R_p||_\alpha = \left(\int_{S_d} |(\theta, s)|^\alpha \Gamma(ds)\right)^{\frac{1}{\alpha}} \tag{12}$$

θ is the d-vector of the portfolio weights and μ^0 is the d-vector of asset return means.

4.1. Efficient set with risky assets.

An efficient portfolio was defined by Markovitz [Mar59] and Sharpe [Sha63] as a portfolio of risky asset which can not achieve greater expected return without increasing risk.

In the absence of a riskless asset, a portfolio P on the efficient frontier is defined as a portfolio solution of the following optimization problem [1] :

$$\min_{\theta \in \mathbb{R}^d} \int_{S_d} |(\theta, s)|^\alpha \Gamma(ds)$$

subject to :

$$\begin{aligned} (\theta, \mu^0) &= E(R_p) \\ (\theta, e) &= 1 \end{aligned}$$

where E(R) is the d-vector of assets expected returns and e denotes an n-vector of ones.

As shown by Press [Pre72] and Arad [Ara75], the efficient set is convex. This means that the efficient frontier is the locus of all convex combinations of any two efficient portfolios.

4.2. Efficient set with one risky and one riskless asset.

In this subsection, we obtain a new and simple relation between the risk and return for efficient portfolio of assets.

Assume now that there exists a risk-free asset F and all investors can borrow or lend unlimited amounts at the riskless rate R_f. The investors satisfy their risk preferences by considering the portfolio P combining θ, $0 < \theta < 1$, of risk-free asset and $(1-\theta)$ of a risky asset I. Then

$$\begin{aligned} E(R_p) &= \theta R_f + (1-\theta) E(R_i) \\ ||R_p||_\alpha^\alpha &= (1-\theta)^\alpha ||R_i||_\alpha^\alpha \end{aligned}$$

[1] this problem will be studied in detail in a forthcoming paper.

which leads to:

$$E(R_p) = R_f + \frac{(E(R_i) - R_f)}{||R_i||_\alpha}||R_p||_\alpha \tag{13}$$

This relation shows that the efficient set in the presence of a riskless asset is represented by a line connecting R_f to the risky asset.

5. Derivation of the CAPM

In this section, we derive a new expression that yields the CAPM in the case of stable multivariate distributions.

If all investors have homogeneous expectations and they all can borrow or lend at the same rate, they will perceive the same efficient set (see figure 1).

In equilibrium, the portfolio of risky assets that an investor tries to combine with the risk-free asset will be identical to the combination held by any other investor[2]. This portfolio must be the tangency portfolio M usually referred to as the market portfolio. The risky assets are held according to their market value weights[3]. All investors will prefer combinations of the risk-free asset and the portfolio M on the same efficient set called the capital market line (see figure 1).

According to (13), this line provides a simple relationship between the risk and return for efficient portfolios of assets. Therefore, the equation of the capital market line will be

$$E(R_p) = R_f + \frac{(E(R_m) - R_f)}{||R_m||_\alpha}||R_p||_\alpha \tag{14}$$

where R_m is the market portfolio return.

We now derive a single period model under the assumptions (1), (2), (3), (4) and (5) described in the introduction.

[2]we note that in equilibrium, all assets must be held, i.e. the excess demand of any asset will be zero.

[3]equal to the market value of each asset divided by the market value of all risky assets.

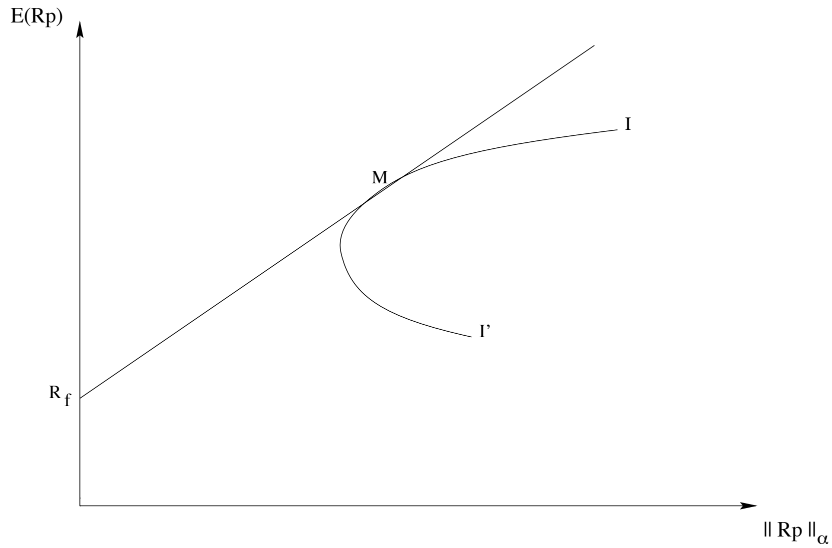

FIGURE 1. the opportunity set provided by combinations of risky asset and the market portfolio

Let us consider a portfolio with θ invested in a risky asset I and $(1-\theta)$ in the market portfolio M. Thus $P = \theta I + (1-\theta)M$. The return of P is then

$$R_p = \theta R_i + (1-\theta)R_m \tag{15}$$

We know from property (2) that since $R_i - E(R_i)$ and $R_m - E(R_m)$ are jointly $S\alpha S$ with $\alpha > 1$, the mean and the scale parameter of R_p are respectively

$$E(R_p) = \theta E(R_i) + (1-\theta)E(R_m) \tag{16}$$

$$||R_p||_\alpha^\alpha = \int_{S_2} |\theta s_1 + (1-\theta)s_2|^\alpha \Gamma(ds) \tag{17}$$

This leads to:

$$\frac{\partial E(R_p)}{\partial \theta} = E(R_i) - E(R_m) \tag{18}$$

$$\frac{\partial ||R_p||_\alpha^\alpha}{\partial \theta} = \alpha ||R_p||_\alpha^{\alpha-1} \frac{\partial ||R_p||_\alpha}{\partial \theta} = \alpha \int_{S_2} (s_1 - s_2)(\theta s_1 + (1-\theta)s_2)^{<\alpha-1>}\Gamma(ds)$$

then

$$\frac{\partial ||R_p||_\alpha}{\partial \theta} = \frac{1}{||R_p||_\alpha^{\alpha-1}} \int_{S_2} (s_1 - s_2)(\theta s_1 + (1-\theta)s_2)^{<\alpha-1>}\Gamma(ds) \tag{19}$$

At point M, $\theta = 0$ and $||R_p||_\alpha = ||R_m||_\alpha$; thus,

$$\begin{aligned} \left.\frac{\partial ||R_p||_\alpha}{\partial \theta}\right|_{\theta=0} &= \frac{1}{||R_m||_\alpha^{\alpha-1}} \int_{S_2} (s_1 - s_2) s_2^{<\alpha-1>} \gamma(ds) \\ &= \frac{1}{||R_m||_\alpha^{\alpha-1}} \left(\int_{S_2} s_1 s_2^{<\alpha-1>} \gamma(ds) - \int_{S_2} s_2^\alpha \gamma(ds) \right) \\ &= \frac{1}{||R_m||_\alpha^{\alpha-1}} \left([R_i, R_m]_\alpha - ||R_m||_\alpha^\alpha \right) \end{aligned}$$

The slope of the risk-return trade-off (curve IMI') evaluated at point M, in market equilibrium, is

$$\frac{\partial E(R_p)}{\partial ||R_p||_\alpha} = \frac{\frac{\partial E(R_p)}{\partial \theta}}{\frac{\partial ||R_p||_\alpha}{\partial \theta}} = \frac{||R_m||_\alpha^{\alpha-1} \left[E(R_i) - E(R_m) \right]}{[R_i, R_m]_\alpha - ||R_m||_\alpha^\alpha} \tag{20}$$

This slope should be equal to the slope of the capital market line given by (14) (see figure 1). Combining (14) and (20) we get

$$\frac{E(R_m) - R_f}{||R_m||_\alpha} = \frac{||R_m||_\alpha^{\alpha-1} \left[E(R_i) - E(R_m) \right]}{[R_i, R_m]_\alpha - ||R_m||_\alpha^\alpha}$$

or

$$(E(R_i) - R_f) = \frac{[R_i, R_m]_\alpha}{||R_m||_\alpha^\alpha} (E(R_m) - R_f) \tag{21}$$

(21) may be written in the form:

$$(E(R_i) - R_f) = \beta_i (E(R_m) - R_f) \tag{22}$$

where

$$\beta_i = \frac{[R_i, R_m]_\alpha}{||R_m||_\alpha^\alpha} \tag{23}$$

The fundamental equation (21) is the generalized equilibrium relationship between risk and return for a given security. It is the generalization of the CAPM, which is usually defined for Gaussian distribution, to the case of stable distribution. We will refer to it as the "stable" CAPM. It may also be called the generalized security market line. The return over the risk free rate, $E(R_i) - R_f$, is called the risk premium for a security I. $E(R_m)-R_f$ is the price of the risk. β_i is the "generalized beta coefficient" which measures the volatility of the security's rate of return relative to changes in the market's rate of return. β_i is interpreted as the quantity of risk.

Remark.

If instead of assumption (2), we assume that asset returns have a joint normal distribution ($\alpha = 2$), then by property (1) of the covariation we obtain

$$(E(R_i) - R_f) = \frac{Cov(R_i, R_m)}{Var(R_m)}(E(R_m) - R_f) \tag{24}$$

which is the well known standard form of the general equilibrium relationship for asset returns often referred to as the Sharpe-Lintner-Mossin form of the CAPM.

Properties of the "stable" CAPM

(1) The only risk which investors will pay a premium to avoid is the covariation risk which is referred to as the systematic risk.
(2) Let $P = \sum_{i=1}^{d} \theta_i Y_i$. The portfolio generalized beta coefficient, denoted β_p, is a linearly weighted combination of the individual asset generalized beta coefficients β_i:

$$\beta_p = \sum_{i=1}^{d} \theta_i \beta_i \tag{25}$$

Proof. *use property (4) of the covariation.*

6. Empirical study

Assume again that the vector R of the per share portfolio follows a multivariate stable distribution with characteristic function

$$E \exp i(\lambda, R) = \exp\left\{ -\int_{S_d} |(\lambda, s)|^\alpha \Gamma(ds) + (\lambda, \mu^0) \right\}. \tag{26}$$

where Γ is symmetric.

This is equivalent to saying that $R - \mu^0$ follows a $S\alpha S$ distribution. The assumption $\alpha > 1$ implies that $\mu^0 = E(R)$.

If the rate of return on any asset is a fair game, the *ex ante* form of the "stable" CAPM given by equation (21) can be transformed into

$$R_i = R_f + (R_m - R_f)\beta_i + \epsilon_i \tag{27}$$

where

- ϵ_i is a random error term independent from R_m with zero mean,
- β_i is as defined in (23).

To estimate the "beta" coefficients, we need to estimate the covariations $[R_i, R_m]_\alpha$. In that view, we use the Rachev-Xin estimator [RX93] of the covariation $[R_i, R_j]_\alpha^{(n)}$ given by:

$$[R_i, R_j]_\alpha^{(n)} = \int_{\Omega_d} s_i(\theta) s_j(\theta)^{<\alpha_n - 1>} d\Phi_n(\theta) \tag{28}$$

where

- (ρ, θ) are the polar coordinates of R;
- $s_i(\theta)$ denotes the i-th component of $s = (s_1, ..., s_d) \in S_d$ in polar coordinates;
- $\Omega_d = [0, \pi]^{d-2} \times [0, 2\pi]$;
- α_n is an estimator of the index α chosen using the family of estimators given by:

$$\alpha_n(k) = \frac{\log 2}{\log(\rho_{n-k+1:n}) - \log(\rho_{n-2k+1:n})} \tag{29}$$

  where $k = (k_n)_{n \geq 1}$ is a sequence of integers satisfying $1 \leq k_n \leq \frac{n}{2}$, $k_n \to \infty$, $\frac{k}{n} \to 0$ as $n \to \infty$ and $\rho_{k:n}$ is the k-th order statistic from $(\rho_1, ..., \rho_n)$. The choice of the optimal value of k_n is based on an empirical rule to be described below;
- Φ_n is an estimator of the distribution function of Γ on Ω_d, obtained through the following steps

$$\Phi_n(\theta) = \varphi_n(\theta)\Phi_n(\Pi), \tag{30}$$

where

$$\Pi = (\pi, ..., \pi, 2\pi) \in \Omega_d, \;\; \theta = (\theta_1, ..., \theta_{d-1}),$$

$$\Phi_n(\Pi) = \frac{k}{n}\rho_{n-k:n}^{\alpha_n} \tag{31}$$

and

$$\varphi_n(\theta) = \frac{1}{k}\sum_{i=1}^{n} \mathbb{1}_{(\Theta_i \leq \theta \;,\; \rho_i \geq \rho_{n-k+1:n})} \tag{32}$$

Rachev and Xin [RX93] showed that:

(1) if $\frac{k}{\log\log n} \to \infty$ as $n \to \infty$ then a.s. $\alpha_n \to \alpha$;
(2) if $\frac{k}{\log n} \to \infty$ as $n \to \infty$ then a.s. $\varphi_n(\theta) \to \varphi(\theta) = \frac{\Phi(\theta)}{\Phi(\Pi)}$.

Under additional conditions, it may be shown that the α-covariation estimator is asymptotically normal.

In the bivariate case ($d = 2$), the α-covariation estimator will take the following simple form

$$[X_1, X_2]_\alpha^{(n)} = \int_0^{2\pi} \cos\theta \sin\theta^{<\alpha_n - 1>} d\Phi_n(\theta) \tag{33}$$

A consistent estimator for the generalized "beta" coefficient for asset i will be

$$\hat{\beta}_i = \frac{[R_i, R_m]_\alpha^{(n)}}{[R_m, R_m]_\alpha^{(n)}} \tag{34}$$

In our study, we have used a sample of nine French stocks chosen from the CAC 40 Index: ACCOR, CAMP. BANCAIRE, CARREFOUR, CR. FONC. FRANCE, GEN. DES EAUX, HAVAS, LEGRAND, MICHELIN, THOMSON CSF. The CAC 40 Index is considered as a "proxy" of the market portfolio.

The data consist of 2059 observations representing the successive differences of the logarithm of daily closing prices for the nine chosen stocks along with the CAC 40 Index, ranging from 09/07/87 to 31/05/95. These data were provided by Crédit Lyonnais.

The optimal k for the estimator of α was selected using a clue given by Mittnik and Rachev ([MR93]). The best value of k is set so as to agree with the estimated marginal index of stability of each asset: we first construct the graph yielding $\alpha_n(k)$ versus $\frac{k}{n}$. We then estimate the marginal index $\hat{\alpha}$. The optimal k^* is the one such that $\alpha_n(k^*) = \hat{\alpha}$ (see figure 2 and 3).

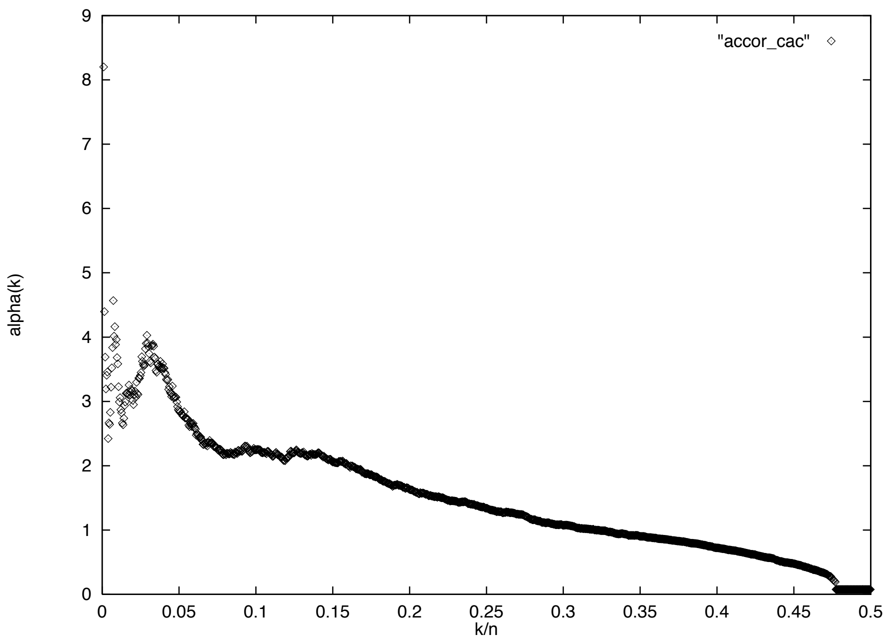

FIGURE 2. The estimator α as a function of k for the ACCOR-CAC40 data.

For the data considered here, the estimated value of the marginal index $\hat{\alpha}$ was found to be near 1.7. The corresponding optimal value of k is given in table 1 below [4]. In order to compare between different kinds of modeling (e.g. Gaussian vs stable non Gaussian), we also computed optimal k's corresponding to "virtual" values of α of 1.3, 1.5 and 2 (this last value corresponds to a "classical" modeling with Gaussian distribution).

[4]values corresponding to the estimated index appears in bold face.

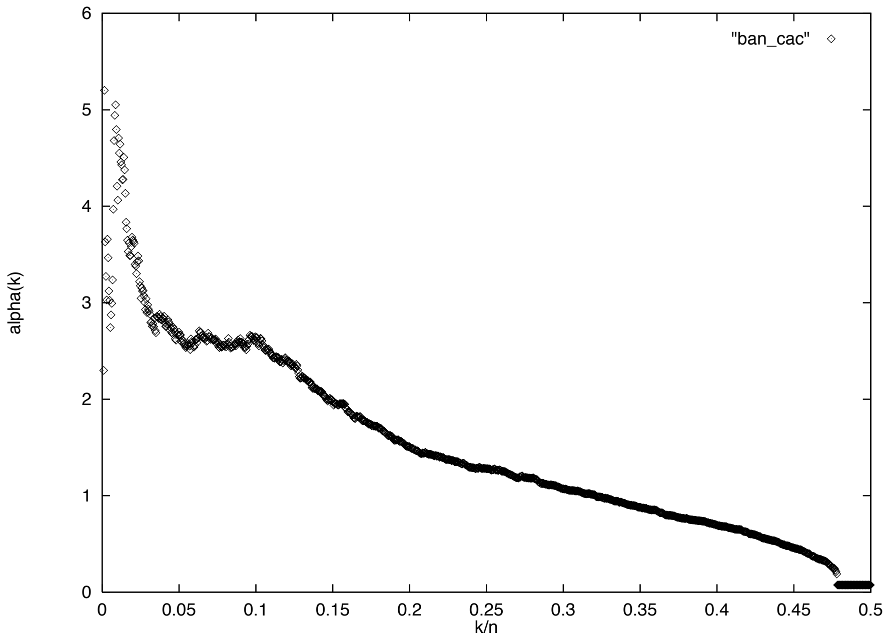

FIGURE 3. The estimator α as a function of k for the BANCAIRE-CAC40 data.

OPTIMAL VALUES OF k				
	Values of marginal α			
Stocks	$\alpha = 2.0$	$\alpha = 1.7$	$\alpha = 1.5$	$\alpha = 1.3$
(ACCOR,CAC40)	329	**391**	455	527
(BANCAIRE,CAC40)	306	**370**	413	490
(GEN.D.EAUX,CAC40)	286	**404**	466	546
(THOM.CSF,CAC40)	247	**328**	409	516
(HAVAS,CAC40)	270	**360**	452	532
(CR.FONC,CAC40)	287	**373**	430	519
(MICHELIN,CAC40)	249	**376**	434	535
(LEGRAND,CAC40)	316	**425**	472	525
(CARREFOUR,CAC40)	307	**380**	457	548

TABLE 1. Optimal values of k for $\alpha = 2,\ 1.7,\ 1.5,\ 1.3$

α-**COVARIATIONS**				
$[X_i, Y_m]_\alpha$	Values of marginal α			
	$\alpha = 2.0$	$\alpha = 1.7$	$\alpha = 1.5$	$\alpha = 1.3$
[ACCOR,CAC40]$_\alpha$	0.000006	**0.000022**	0.000055	0.000144
[BANCAIRE,CAC40]$_\alpha$	0.000007	**0.000029**	0.000074	0.000192
[GEN.D.EAUX,CAC40]$_\alpha$	0.000006	**0.000022**	0.000056	0.000145
[THOM.CSF,CAC40]$_\alpha$	0.000007	**0.000028**	0.000073	0.000183
[HAVAS,CAC40]$_\alpha$	0.000006	**0.000024**	0.000060	0.000153
[CR.FONC,CAC40]$_\alpha$	0.000006	**0.000023**	0.000057	0.000150
[MICHELIN,CAC40]$_\alpha$	0.000007	**0.000028**	0.000071	0.000150
[LEGRAND,CAC40]$_\alpha$	0.000005	**0.000018**	0.000048	0.000129
[CARREFOUR,CAC40]$_\alpha$	0.000006	**0.000021**	0.000053	0.000134
[CAC40,CAC40]$_\alpha$	0.000006	**0.000020**	0.000049	0.000123

TABLE 2. Estimated values of α-covariations

Table 2 and table 3 show that the covariations and the "beta" coefficients increase as α tends to 1. The average increase of the "beta" coefficients is about 16% when α decreases from 2 to 1.7, with a maximum increase of 28% for Cie BANCAIRE.

To stress the significance of this result, let us take the example of the THOMSON CSF and the ACCOR stocks. In the classical Gaussian frame, the estimated "beta" coefficients are respectively 1.17 and 1.00. If we use the more realistic value $\alpha = 1.7$, they increase up to respectively 1.40 and 1.10, i.e. increases of 23% and 10%. In other words, the sensibilities of the THOMSON CSF and ACCOR stocks to the market portfolio are markedly greater than the ones estimated with the "Gaussian" CAPM. The same phenomenon is observed for all others stocks but with different variations of the coefficient. This has two important consequences: when building an index replicated portfolio with $\alpha = 1.7$, the proportion of each stock will not be the same as in the "Gaussian" case. Assuming wrongly that asset returns have a joint normal distribution thus results both in an underevaluation of the "beta"of the stock (and

"BETA" COEFFICIENTS				
Stocks	Values of marginal α			
	$\alpha = 2.0$	$\alpha = 1.7$	$\alpha = 1.5$	$\alpha = 1.3$
ACCOR	1.00	**1.10**	1.13	1.17
BANCAIRE	1.17	**1.45**	1.51	1.56
GEN.D.EAUX	1.00	**1.10**	1.15	1.18
THOM.CSF	1.17	**1.40**	1.49	1.49
HAVAS	1.00	**1.20**	1.23	1.24
CR.FONC	1.00	**1.15**	1.17	1.21
MICHELIN	1.17	**1.40**	1.45	1.48
LEGRAND	0.83	**0.90**	1.00	1.05
CARREFOUR	1.00	**1.05**	1.08	1.10

TABLE 3. Estimated values of generalized "beta" coefficients

consequently of the risk premium) and in a non optimal portfolio allocation.

To summarize, the classical CAPM based on the mean-variance approach is generally misleading because it discards important information about the risk structure of different investment opportunities. This means that the MV-efficient portfolio is not efficient in the "stable" CAPM context. It is thus necessary to generalize the notion of efficiency from MV-efficiency to stable-efficiency.

7. CONCLUSION

The "stable" CAPM derived here represents a generalized model for evaluating both the risk and the expected return of alternative portfolios. It offers a valuable alternative for measuring the risk and differs from the classical one by its ability to deal with high risks induced by strong variations of markets. It should then be a useful and convenient tool for fund managers and investors who seek to maximize their trade-off between risk and return. Indeed, the Gaussian model of portfolio optimization underevaluates the real risk, because it is based on a theoretical framework not well fitted to the real market. On the contrary, the generalized CAPM allows the investors and fund managers to adequately price the risk in the real world.

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